Lecture 10
Subgame-perfect Equilibrium
&
Applications

14.12 Game Theory

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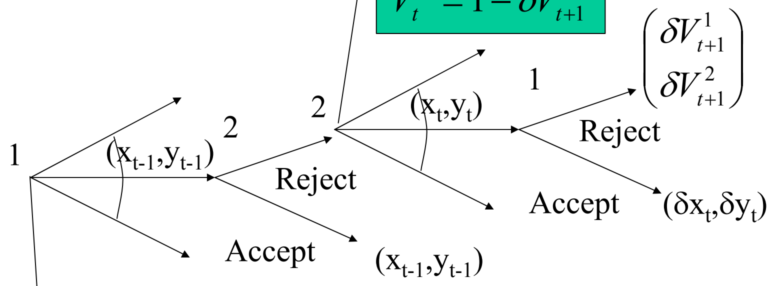
Road Map

1. Bargaining – Backwards induction completed
2. Subgame-perfect Equilibrium
    1. Motivation
    2. What is a subgame?
    3. Definition
    4. Example
3. Quiz
4. More Bargaining – Subgame-Perfect Eqm.

The share of i at t+1 = V_{t+1}^{i}

$$V_t^1 = \delta V_{t+1}^1$$
$$V_t^2 = 1 - \delta V_{t+1}^1$$



$$V_{t-1}^1 = 1 - \delta V_t^2 \qquad = 1 - \delta\left(1 - \delta V_{t+1}^1\right) \quad = 1 - \delta + \delta^2 V_{t+1}^1$$
$$V_{t-1}^2 = \delta V_{t+1}^2$$

$$V_{2n-2k-1}^1 = 1 - \delta + \delta^2 \left[1 - \delta + \delta^2 V_{2n-2k+3}^1 \right]$$
$$= 1 - \delta + \delta^2(1-\delta) + \delta^4 \left[1 - \delta + \delta^2 V_{2n-2k+5}^1 \right]$$
$$= 1 - \delta + \delta^2(1-\delta) + \delta^4(1-\delta) + \delta^6 V_{2n-2k+5}^1$$

.
.
.

$$= (1-\delta)\left(1 + \delta^2 + \delta^4 + \cdots + \delta^{2k}\right)$$
$$= \frac{1-\delta^{2k+1}}{1+\delta}$$

$$n \longrightarrow \infty$$

$$t = 2n - 2k - 1$$

$$x_t = \frac{1-\delta^{2k+1}}{1+\delta} = \frac{1-\delta^{2n-t}}{1+\delta} \xrightarrow{n\to\infty} \frac{1}{1+\delta}$$

Timeline – ∞ period

$T = \{1,2,\ldots, n-1,n,\ldots\}$

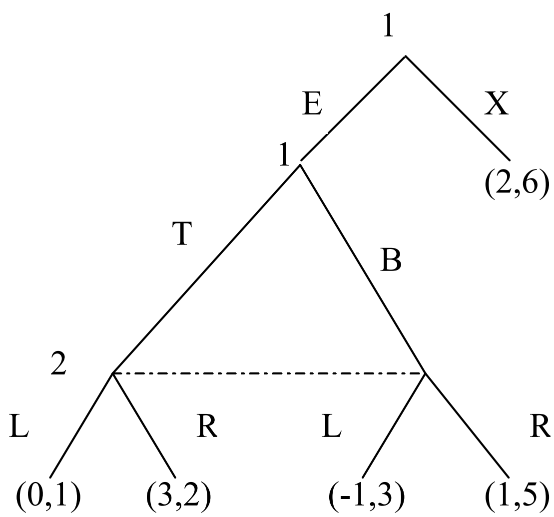
If t is odd,

- Player 1 offers some (x_t,y_t),
- Player 2 Accept or Rejects the offer
- If the offer is Accepted, the game ends yielding $\delta^t(x_t,y_t)$,
- Otherwise, we proceed to date t+1.

If t is even

- Player 2 offers some (x_t,y_t),
- Player 1 Accept or Rejects the offer
- If the offer is Accepted, the game ends yielding payoff (x_t,y_t),
- Otherwise, we proceed to date t+1.

A game



Backward induction

- Can be applied only in perfect information games of finite horizon.

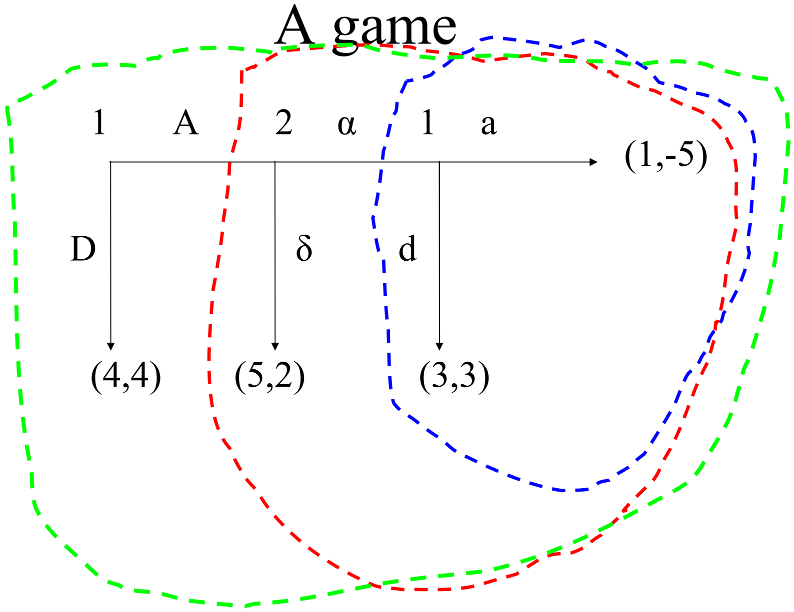
How can we extend this notion to infinite horizon games, or to games with imperfect information?

A subgame

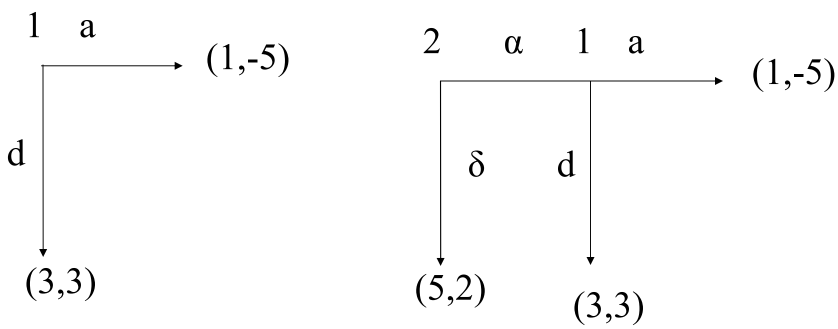
A *subgame* is part of a game that can be considered as a game itself.

- It must have a unique starting point;
- It must contain all the nodes that follow the starting node;
- If a node is in a subgame, the entire information set that contains the node must be in the subgame.

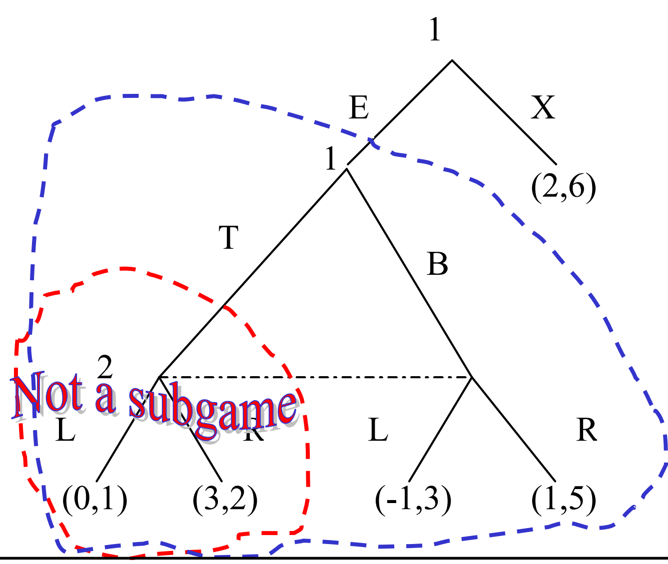
A game



And its subgames



A game

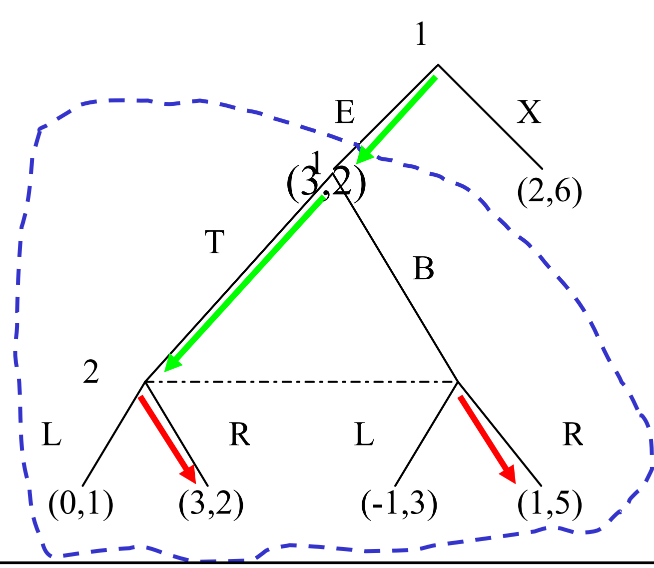


Definitions

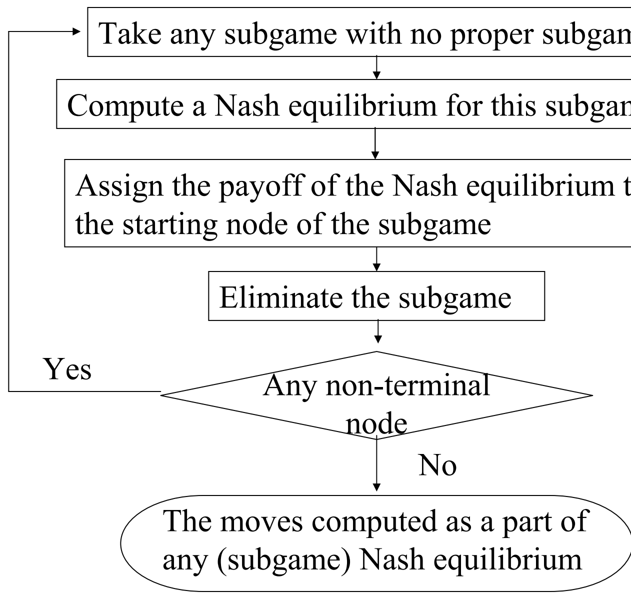
A *substrategy* is the restriction of a strategy to a subgame.

A subgame-perfect Nash equilibrium is a Nash equilibrium whose substrategy profile is a Nash equilibrium at each subgame.

Example



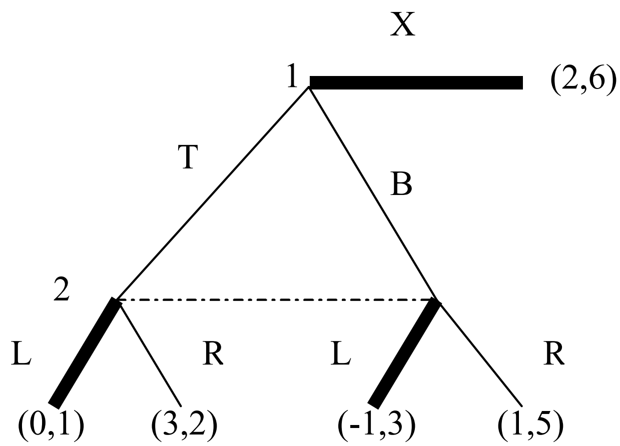
A "Backward-Induction-like" method



Theorem

In a finite, perfect-information game, the set of subgame-perfect equilibria is the set of strategy profiles that are computed via backward induction.

A subgame-perfect equilibrium?



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$$n \longrightarrow \infty$$

t = 2n – 2k – 1

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A SPE: At each t,

- proposer offers δ/(1+δ) to the other
- and keeps 1/(1+δ) for himself;
- responder accepts an offer iff
- she gets at least δ/(1+δ) .

Bank Run